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Department of Mathematics
2-marks Semester IV
Unit I Probability and Random Variable

1.. Define Random Variable (RV).

A random variable is a function X: S $\rightarrow$ R that assigns a real number X(S) to every element $s \in S$, where S is the sample space corresponding to a random experiment E.

Ex: Consider an experiment of tossing an unbiased coin twice. The outcomes of the experiment are HH, HT, TH,TT.let X denote the number of heads turning up. Then X has the values 2,1,1,0. Here X is a random variable which assigns a real number to every outcome of a random experiment.

2. Define Discrete Random Variable.

If X is a random variable which can take a finite number or countably infinite number of pvalues, X is called a discrete RV.

Ex. Let X represent the sum of the numbers on the 2 dice, when two dice are trown.

3. Define Continuous Random Variable.

If X is a random variable which can take all values (i.e., infinite number of values) in an interval, then X is called a continuous RV.

Ex. The time taken by a lady who speaks over a telephone.

4. Define One-dimensional Random Variables.

If a random variable X takes on single value corresponding to each outcome of the experiment, then the random variable is called one-dimensional random variables.it is also called as scalar valued RVs.

Ex:

In coin tossing experiment, if we assume the random variable to be appearance of tail, then the sample space is{H,T} and the random variable is{1,0}.which is an one-dimensional random variables.

5. State the Properties of expectation.

If X and Y are random variables and a,b are constants, then

1.E(a)=a

Proof:

$$E(X)=\sum_{i=1}^{n} x_i p_i$$

$$E(a)= \sum_{i=1}^{n} ap_i = a\sum_{i=1}^{n} p_i = a(1) \ (\because \sum_{i=1}^{n} p_i = 1)$$

E(a) =a

2.E(aX)=aE(X)

Proof:

$$E(X)=\sum_{i=1}^{n} x_i p_i$$

$$E(aX)= \sum_{i=1}^{n} ax_i p_i = a\sum_{i=1}^{n} x_i p_i = aE(X)$$

3.E(aX+b)=aE(X)+b

Proof:

$$E(X)=\sum_{i=1}^{n} x_i p_i$$

E(aX+b)= $\sum_{i=1}^{n}(ax_i+b)p_i=\sum_{i=1}^{n}(ax_i)p_i+\sum_{i=1}^{n}bp_i=a\sum_{i=1}^{n}x_i p_i+b\sum_{i=1}^{n}p_i$

E(aX+b) =a E(X)+b $\{\because \sum_{i=1}^{n}p_i=1\}$

4.E(X+Y)=E(X)+E(Y)

5. E(XY)=E(X).E(Y), if X and Y are random variables.

6. E(X- $\overline{X}$)=E(X)- $\overline{X}$ = $\overline{X}$ - $\overline{X}$ =0

6. A RV X has the following probability function

Values of X	0	1	2	3	4	5	6	7	8
P(x)	a	3a	5a	7a	9a	11a	13a	15a	17a

1) Determine the value of a.

2) Find P(X<3), P(X$\geq$3), P(0<X<5).

Solution:

1) We know that $\sum_{x}P(x)=1$

a+3a+5a+7a+9a+11a+13a+15a+17a=1

81a = 1

a = 1/81

2) P(X<3) = P(X=0) + P(X=1) +P(X=2)

= a+3a+5a

=9a =9/81=1/9

P(X$\geq$3) = 1- P(X<3) =1-1/9 = 8/9

P(0<X<5) = P(X=1) + P(X=2) + P(X=3) +P(X=4)

= 3a+5a+7a +9a =24a= 24/81

7. If X is a continuous RV whose PDF is given by

$$f(x)=\begin{cases}c(4x-2x^2), 0<x<2\\ 0, \qquad otherwise\end{cases}$$

Find c.and mean of X.

Solution:

We know that $\int_{-\infty}^{\infty}f(x)dx=1$

$$\int_{0}^{2}c(4x-2x^2)dx=1$$

c = 3/8

$$E(X)=\int_{-\infty}^{\infty}xf(x)dx=\int_{0}^{2}\frac{3}{8}x(4x-2x^2)dx=\frac{8}{3}$$

8. A continuous RV X that can assume any value between x = 2 and x = 5 has a density function given by $f(x)=k(1+x)$.**Fnd *P(X<4).***

Solution:

We know that $\int_{-\infty}^{\infty}f(x)dx=1$

$$\int_{2}^{5}k(1+x)dx=1$$

k = 2/27

$$P(X<4)=\int_{2}^{4}\frac{2}{27}(1+x)dx=\frac{16}{27}$$

9. A RV X has the density function

$$f(x)=\begin{cases} k\dfrac{1}{1+x^2}, -\infty < x < \infty \\ 0, \quad otherwise \end{cases}$$ **. Find k .**

Solution:

We know that $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\int_{-\infty}^{\infty} k\frac{1}{1+x^2}dx = 1$$

$$k(\tan^{-1}x)_{-\infty}^{\infty} = 1$$

$$k\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = 1$$

$$\therefore k = \frac{1}{\pi}$$

10. If the p.d.f of a RV .X is given by $f(x)=\begin{cases} \frac{1}{4}, -2 < X < 2 \\ 0, elsewhere \end{cases}$ **.Find** $P[|X|>1]$.

Answer:

$$P[|X|>1] = 1-P[|X|<1] = 1-\int_{-1}^{1}\frac{1}{4}dx = 1-\frac{1}{4}[1+1] = 1-\frac{1}{2} = \frac{1}{2}$$

11. If the pdf of a RV X is $f(x)=\frac{x}{2}$ **in** $0 \le x \le 2$, find $P[X > 1.5 / X > 1]$

Answer:

$$P[X > 1.5/X > 1] = \frac{p[X > 1.5]}{P[X > 1]} = \frac{\int_{1.5}^{2}\frac{x}{2}dx}{\int_{1}^{2}\frac{x}{2}dx} = \frac{4-2.25}{4-1} = 0.5833$$

12. Determine the Binomial distribution whose mean is 9 and whose SD is 3/2

Ans : np = 9 and npq = 9/4 $\therefore$ q = $\frac{npq}{np} = \frac{1}{4}$

$\Rightarrow$ p = 1- q = $\frac{3}{4}$. np = 9 $\Rightarrow$ n = $9\left(\frac{4}{3}\right)$=12.

P [x = r] = $12C_r\left(\frac{3}{4}\right)^r\left(\frac{1}{4}\right)^{12-r}$, r = 0,1,2,…,12.

13. Find the M.G.F of a Binomial distribution

$$M_x(t) = \sum_{i=0}^{n} e^{tx}\, {}_nC_x p^x q^{n-x} = \sum_{x=0}^{n} {}_nC_r (pe^t)^x q^{n-x} = (q+pe^t)^n$$

14. The mean and variance of the Binomial distribution are 4 and 3 respectively. Find P(X=0).

Ans :

mean = np = 4, Variance = npq = 3

$q = \frac{3}{4}$, $\quad p = 1 - \frac{3}{4} = \frac{1}{4}$, $\quad$ np = 4 $\Rightarrow$ n = 16

P (X=0) $= {}_nC_0 p^0 q^{n-0} = 16C_0 p^0 q^{16-0} = \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^{16} = \left(\frac{3}{4}\right)^{16}$

15. For a Binomial distribution mean is 6 and standard deviation is $\sqrt{2}$. Find the first two terms of the distribution.

Ans : Given np = 6, npq $= \left(\sqrt{2}\right)^2 = 2$

$q = \frac{2}{6} = \frac{1}{3}$, $\quad p = 1 - q = \frac{2}{3}$, np = 6 $\Rightarrow n\left(\frac{2}{3}\right) = 6 \Rightarrow n = 9$

P (X= 0) $= {}_nC_0 p^0 q^{n-0} = 9C_0 \left(\frac{2}{3}\right)^0 \left(\frac{1}{3}\right)^{9-0} = \left(\frac{1}{3}\right)^9$

P (X=1) $= {}_nC_1 p^1 q^{n-1} = 9\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^8 = 6\left(\frac{1}{3}\right)^8$

16. The mean and variance of a binomial variate are 4 and $\frac{4}{3}$ respectively,

find $P[X \geq 1]$.

Ans : np = 4, npq $= \frac{4}{3} \Rightarrow q = \frac{1}{3}, p = \frac{2}{3}$

$P[X \geq 1]$= 1- P[X<1] $\quad$ = 1 – P[X = 0] =1-$\left(\frac{1}{3}\right)^6$ = 0.9986

17. For a R.V X, $M_x(t) = \frac{1}{81}(e^t + 2)^4$. Find $P(X \leq 2)$.

Sol: Given $M_x(t) = \frac{1}{81}(e^t + 2)^4 = \left(\frac{e^t}{3} + \frac{2}{3}\right)^4$ ------------------- (1)

For Binomial Distribution, $M_x(t) = (q + pe^t)^n$. ------------------------- (2)

Comparing (1)&(2),

$\therefore n = 4, \; q = \frac{2}{3}, \; p = \frac{1}{3}.$

$$P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2) = 4C_0\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^4 + 4C_1\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^3 + 4C_2\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)$$

$$= \frac{1}{81}(16 + 32 + 24) = \frac{72}{81} = 0.8889.$$

18. If a R.V X takes the values -1,0,1 with equal probability find the M.G.F of X.

Sol: P[X= -1]=1/3 , P[X=0]=1/3, P[X=1]=1/3

$$M_x(t) = \sum_x e^{tx} P(X = x) = \frac{1}{3}e^{-t} + \frac{1}{3} + \frac{1}{3}e^t = \frac{1}{3}(1 + e^t + e^{-t}).$$

19. A die is thrown 3 times. If getting a 6 is considered as success find the probability of atleast 2 success.

Sol: $p = \frac{1}{6}, q = \frac{5}{6} n = 3.$

P(at least 2 success)=P(X$\geq$2) = P(X=2)+P(X=3)

$$= 3C_2\left(\frac{1}{6}\right)^2\frac{5}{6} + 3C_3\left(\frac{1}{6}\right)^3 = \frac{2}{27}.$$

20. Find p for a Binimial variate X if n=6,and 9P(X=4)=P(X=2).

Sol: $9P(X=4)=P(X=2) \Rightarrow 9\left({}_6C_4 p^4 q^2\right) = {}_6C_2 p^2 q^4$

$$\Rightarrow 9p^2 = q^2 = (1-p)^2 \therefore 8p^2 + 2p - 1 = 0$$

$$\therefore p = \frac{1}{4}\left(\because p \neq -\frac{1}{2}\right)$$

21. Comment on the following

"The mean of a BD is 3 and variance is 4"

For B.D, Variance< mean

$\therefore$ The given statement is wrongs

22. Define poisson distribution

A discrete RV X is said to follow Poisson Distribution with parameter λ if its probability mass function is p(x) = $\frac{e^{-\lambda}\lambda^x}{x!}$, x = 0, 1,2, …… ∞

23. If X is a Poisson variate such that P(X=2)=9P (X = 4) + 90 P(X=6), find the variance

Ans : P [X = x] = $\frac{e^{-\lambda}\lambda^x}{x!}$

Given P(X=2)=9P (X = 4) + 90 P(X=6)

$$\therefore \frac{e^{-\lambda}\lambda^2}{2!} = 9\frac{e^{-\lambda}\lambda^4}{4!} + 90\frac{e^{-\lambda}\lambda^6}{6!}$$

$$\Rightarrow \frac{1}{2} = \frac{9}{24}\lambda^2 + \frac{90}{720}\lambda^4 \Rightarrow \lambda^4 + 3\lambda^2 - 4 = 0$$

$$\Rightarrow (\lambda^2+4)(\lambda^2-1) = 0$$

$$\Rightarrow \lambda^2 = -4 \quad \text{or} \quad \lambda^2 = 1$$

hence $\lambda = 1 [\because \lambda^2 \neq -4]$ Variance=1.

24. It is known that 5% of the books bound at a certain bindery have defective bindings. find the probability that 2 of 100 books bound by this bindery will have defective bindings.

Ans : Let X denote the number of defective bindings.

p = $\frac{5}{100}$ n = 100 $\therefore \lambda = np = 5$

P [X =2] = $\frac{e^{-\lambda}\lambda^2}{2!} = \frac{e^{-5}(25)}{2}$ = 0.084

25. Find λ, if X follows Poisson Distribution such that P(X=2)=3P(X=3).

Sol: P(X=2)=3P(X=3) $\Rightarrow \frac{e^{-\lambda}\lambda^2}{2!} = \frac{3e^{-\lambda}\lambda^3}{3!} \Rightarrow \frac{1}{2} = \frac{3\lambda}{6} \Rightarrow \lambda = 1.$

26. If X is a Poisson variate such that $P(X=1) = \frac{3}{10}$ and $P(X=2) = \frac{1}{5}$. Find $P(X=0)$ and $P(X=3)$.

Sol: $P(X=1) = \frac{3}{10} \Rightarrow \frac{e^{-\lambda}\lambda}{1} = \frac{3}{10}$ ………..(1)

$P(X=2) = \frac{1}{5} \Rightarrow \frac{e^{-\lambda}\lambda^2}{2} = \frac{1}{5}$ …………….(2)

$$\frac{(2)}{(1)} \Rightarrow \frac{\lambda}{2} = \frac{10}{15} \Rightarrow \lambda = \frac{4}{3} \qquad \therefore P(X=0) = \frac{e^{-\frac{4}{3}}\left(\frac{4}{3}\right)^0}{0!} = 0.2636$$

$$\therefore P(X=3) = \frac{e^{-\frac{4}{3}}\left(\frac{4}{3}\right)^3}{3!}$$

27. For a Poisson Variate X , $E(X^2) = 6$ .What is E(X).

Sol: $\lambda^2 + \lambda = 6 \Rightarrow \lambda^2 + \lambda - 6 = 0 \Rightarrow \lambda = 2, -3.$

But $\lambda > 0 \therefore \lambda = 2$ Hence $E(X) = \lambda = 2$

28. A Certain Blood Group type can be find only in 0.05% of the people. If the population of a randomly selected group is 3000.What is the Probability that atleast a people in the group have this rare blood group.

Sol: p=0.05% =0.0005 n=3000 $\therefore \lambda = np = 1.5$

$$P(X \ge 2) = 1 - P(X < 2) = 1 - P(X=0) - P(X=1)$$

$$= 1 - e^{-1.5}\left[1 + \frac{1.5}{1}\right] = 0.4422.$$

29. If X is a poisson variate with mean λ show that E[X^2] = λ E[X+1]

E[X^2] = $\lambda^2 + \lambda$

E (X+1) = E[X] + 1 $\therefore E[X^2] = \lambda E[X+1]$

30. Find the M.G.F of Poisson Distribution.

Ans :

$$M_X(t) = \sum_{x=0}^{\infty} e^{tx}\frac{\lambda^x e^{-\lambda}}{x!} = \sum_{x=0}^{\infty}\frac{(e^t\lambda)^x e^{-\lambda}}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{(e^t\lambda)^x}{x!} = e^{-\lambda}e^{\lambda e^t} = e^{\lambda(e^t-1)}$$

31. A discrete RV X has M.G.F $M_x(t) = e^{2(e^t-1)}$. Find E (X), var (X) and P(X=0)

Ans : $M_X(t) = e^{2(e^t-1)} \Rightarrow$ X follows Poisson Distribution $\therefore \lambda = 2$

Mean = E (X) = $\lambda = 2$ Var (X) = $\lambda = 2$

$$P[X=0] = \frac{e^{-\lambda}\lambda^0}{0!} = \frac{e^{-2}2^0}{0!} = e^{-2}$$

32. If the probability that a target is destroyed on any one shot is 0.5, what is the probability that it would be destroyed on 6th attempt?

Ans : Given p = 0.5 q = 0.5

By Geometric distribution

P [X =x] = q^x p, x = 0,1,2………

since the target is destroyed on 6th attempt x = 5

$\therefore$ Required probability = q^x p = $(0.5)^6$ = 0.0157

33. Find the M.G.F of Geometric distribution

$$M_X(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx}q^x p = p\sum_{x=0}^{\infty}(qe^t)^x$$

$$= p[1-qe^t]^{-1} = \frac{p}{1-qe^t}$$

34. Find the mean and variance of the distribution $P[X=x]=2^{-x}$, x=1,2,3….

Solution:

$$P[X=x]= \frac{1}{2^x} = \left(\frac{1}{2}\right)^{x-1}\frac{1}{2}, x = 1,2,3.....$$

$$\therefore p = \frac{1}{2} \text{ and } q = \frac{1}{2}$$

$$Mean = \frac{q}{p} = 1; \; Variance = \frac{q}{p^2} = 2$$

35. Find the expected value and the variance of the number of times one must throw a die until the outcome 1 has occurred 4 times.

Solution:

X follows the negative bionomial distribution with parameter r = 4 and p=1/6

E(X)= mean = r P = rqQ = r (1-p) (1/p)= 20. (p=1/Q and q=P/Q)

Variance = rPQ = $r(1-p)/p^2$ = 120.

36. If a boy is throwing stones at a target, what is the probability that his 10th throw is his 5th hit, if the probability of hitting the target at any trial is ½?

Solution:

Since 10th throw should result in the 5th successes ,the first 9 throws ought to have resulted in 4 successes and 5 faliures.

n = 5,r = 5, $p = \frac{1}{2} = q$

$\therefore$ Required probability = P(X=5)= $(5+5-1)C_5\ (1/2)^5\ (1/2)^5$

$= 9C_4\ (1/2^{10}) = 0.123$

37. Find the MGF of a uniform distribution in (a, b)?

Ans :

$$M_X(t) = \frac{1}{b-a}\int_a^b e^{tx}dx \qquad = \frac{e^{bt} - e^{at}}{(b-a)t}$$

38. Find the MGF of a RV X which is uniformly distributed over (-2, 3)

$$M_X(t) = \frac{1}{5}\int_{-2}^{3} e^{tx}dx = \frac{e^{3t} - e^{-2t}}{5t} \; for \; t \neq 0$$

39. The M.G.F of a R.V X is of the form $M_x(t) = (0.4e^t + 0.6)^8$ what is the M.G.F of the R.V, Y = 3X +2

$M_Y(t) = e^{2t}M_X(t) = e^{2t}((0.4)\ e^{3t} + 0.6)^8$

40. If X is uniformly distributed with mean 1 and variance $\frac{4}{3}$ find P(X<0)

Ans : Let X follows uniform distribution in (a,b)

mean = $\frac{b+a}{2} = 1$ and Variance = $\frac{(b-a)^2}{12} = \frac{4}{3}$

$\therefore$ a+b = 2 $(b-a)^2 = 16 \Rightarrow b - a = \pm 4$

Solving we get a=-1 b=3

$\therefore$ f(X) = $\frac{1}{4}$, - 1 < x < 3

$\therefore$ P[X< 0] = $\int_{-1}^{0} f(x)dx = \frac{1}{4}$

41. A RV X has a uniform distribution over (-4, 4) compute $P(|X|>2)$

Ans :

$$f(x) = \begin{cases} \frac{1}{8}, & -4 < x < 4 \\ 0 & \text{other wise} \end{cases}$$

$$P(|X|>2) = 1 - P(|X| \le 2) = 1 - P(-2 < X < 2) = 1 - \int_{-2}^{2} f(x)dx = 1 - \frac{4}{8} = \frac{1}{2}$$

42. If X is Uniformly distributed in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$**.Find the p.d.f of Y=tan X.**

Sol: $f_X(x) = \frac{1}{\pi}$; $X = \tan^{-1} Y \Rightarrow \frac{dx}{dy} = \frac{1}{1+y^2}$.

$$\therefore f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| \Rightarrow f_Y(y) = \frac{1}{\pi(1+y^2)}, \; -\infty < y < \infty$$

43. If X is uniformly distributed in (-1,1).Find the p.d.f of $y = \sin\frac{\pi x}{2}$**.**

Sol:

$$f_X(x) = \begin{cases} \frac{1}{2}, -1 < x < 1 \\ 0, otherwise \end{cases},$$

$$x = \frac{2\sin^{-1} y}{\pi} \Rightarrow \frac{dx}{dy} = \frac{2}{\pi}\frac{1}{\sqrt{1-y^2}} \quad \text{for} \quad -1 \le y \le 1$$

$$f_Y(y) = \frac{1}{2}\left[\frac{2}{\pi}\frac{1}{\sqrt{1-y^2}}\right] \Rightarrow f_Y(y) = \frac{2}{\pi}\frac{1}{\sqrt{1-y^2}} \quad \text{for} \quad -1 \le y \le 1$$

44.The time (in hours) required to repair a machine is exponentially distributed with parameter $\lambda = \frac{1}{2}$ **what is the probability that a repair takes at least 10 hours given that its duration exceeds 9 hours?**

Ans :

Let X be the RV which represents the time to repair machine.

P[X≥ 10/ X ≥ 9] = P [X≥ 1] (by memory less property]

$= \int_{1}^{\infty} \frac{1}{2} e^{-x/2} dx = 0.6065$

45. The time (in hours) required to repair a machine is exponentially distributed with parameter $\lambda = \frac{1}{3}$ **what is the probability that the repair time exceeds 3 hours?**

Ans : X – represents the time to repair the machine

$$\therefore f(x) = \frac{1}{3}e^{-x/3} > 0$$

P (X > 3) = $\int_3^\infty \frac{1}{3}e^{-x/3}dx = e^{-1} = 0.3679$

46.Find the M.G.F of an exponential distribution with parameter λ.

Sol: $M_x(t) = \lambda\int_0^\infty e^{tx}e^{-\lambda x}dx = \lambda\int_0^\infty e^{-(\lambda-t)x}dx = \frac{\lambda}{\lambda-t}$

47. State the memory less property of the exponential distribution.

Soln: If X is exponential distributed with parameter λ then

$P(X > s+t / X > s) = P(X > t)$ for s , t >0

48. If X has a exponential distribution with parameter λ,find the p.d.f of Y=log X.

Sol: Y=log X$\Rightarrow e^y = x \Rightarrow \frac{dx}{dy} = e^y$

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|. \Rightarrow f_Y(y) = e^y\lambda e^{-\lambda e^y}.$$

49. If X has Exponential Distribution with parameter 1,find the p.d.f of $Y = \sqrt{X}$.

Sol: $Y = \sqrt{X} \Rightarrow X = Y^2$ $f_X(x) = e^{-x}, x > 0.$

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = 2ye^{-x} = 2ye^{-y^2}, y > 0.$$

50 . Write the M.G.F of Gamma distribution

$M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx}f(x)dx$

$= \frac{\lambda^\gamma}{\Gamma\gamma}\int_0^\infty e^{-(\lambda-t)x}x^{\gamma-1}dx = \frac{\lambda^\gamma}{\Gamma\gamma}\frac{\Gamma\gamma}{(\lambda-t)^\gamma}$

$$\therefore Mx(t) = \left(1-\frac{t}{\lambda}\right)^{-\gamma}$$

51. Define Normal distribution

A normal distribution is a continuous distribution given by

$y = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}$ where X is a continuous normal variate distributed with density function $f(X) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}$ with mean μ and standard deviation σ

52. What are the properties of Normal distribution

(i) The normal curve is symmetrical when p = q or p $\approx$ q.

(ii) The normal curve is single peaked curve.

(iii) The normal curve is asymptotic to x-axis as y decreases rapidly when x increases numerically.

(iv) The mean, median and mode coincide and lower and upper quartile are equidistant from the median.

(v) The curve is completely specified by mean and standard deviation along with the value y_o.

53. Write any four properties of normal distribution.

Sol: (1) The curve is Bell shaped

(2) Mean, Median, Mode coincide

(3) All odd moments vanish

(4) x - axis is an asymptote of the normal curve

54. If X is a Normal variate with mean30 and SD 5.Find P [26<X<40].

Sol: P [26<X<40] =P $[-0.8 \le Z \le 2]$ where $Z = \frac{X-30}{5}$ $\left\{\because Z = \frac{X-\mu}{\sigma}\right\}$

$= P[0 \le Z \le 0.8] + P[0 \le Z \le 2]$

= 0.2881+0.4772

=0.7653.

55. If X is normally distributed RV with mean 12 and SD 4.Find P [X≤ 20].

Sol: $P[X \le 20] = P[Z \le 2]$ where $Z = \frac{X-12}{4}$ $\left\{\because Z = \frac{X-\mu}{\sigma}\right\}$

$= P[-\infty \le Z \le 0] + P[0 \le Z \le 2]$

=0.5+0.4772

=0.9772.

56. **If X is a N(2,3),find** $P\left[Y \ge \frac{3}{2}\right]$ **where Y+1=X.**

Answer:

$$P\left[Y \ge \frac{3}{2}\right] = P\left[X - 1 \ge \frac{3}{2}\right] = P[X \ge 2.5] = P[Z \ge 0.17] = 0.5 - P[0 \le Z \le 0.17]$$

$$= 0.5 - 0.0675 = 0.4325$$

57.. If X is a RV with p.d.f $f(x) = \frac{x}{12}$ **in 1 < x < 5 and =0, otherwise.Find the p.d.f of Y=2X-3.**

Sol: Y=2X-3 $\Rightarrow \frac{dx}{dy} = \frac{1}{2}$

$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = \frac{y+3}{4}$,. in -1<y<7.

58.. If X is a Normal R.V with mean zero and variance σ^2 **Find the p.d.f of** $Y = e^X$ **.**

Sol: $Y = e^X \Rightarrow \log Y = X \Rightarrow \frac{dx}{dy} = \frac{1}{y}$

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = \frac{1}{y} f_X(\log y)$$

$$= \frac{1}{\sigma y\sqrt{2\pi}} \exp\left(-\left(\log y - \mu^2\right)/2\sigma^2\right)$$

59. If X has the p.d.f $f(x) = \begin{cases} x, 0 < x < 1 \\ 0, otherwise \end{cases}$ **find the p.d.f of** $Y = 8X^3$**.**

Sol: $Y = 8X^3 \Rightarrow X = \frac{1}{2}Y^{\frac{1}{3}} \quad \Rightarrow \frac{dx}{dy} = \frac{1}{6}y^{-\frac{2}{3}}$

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = (x)\left(\frac{1}{6}Y^{-\frac{2}{3}}\right) = \frac{1}{12}Y^{-\frac{1}{3}}, Y > 0$$

60. The p.d.f of a R.V X is $f(x) = 2x, 0 < x < 1$.**Find the p.d.f of** $Y = 3X + 1$.

Sol: $Y = 3X + 1 \Rightarrow X = \frac{Y-1}{3}$

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = 2x\frac{1}{3} = 2\left(\frac{y-1}{3}\right)\left(\frac{1}{3}\right) = \frac{2(y-1)}{9}.,1 < y < 4.$$

Moment generating functions

1. Define n^{th} **Moments about Origin**

The n^{th} moment about origin of a RV X is defined as the expected value of the n^{th} power of X.

For discrete RV X, the n^{th} moment is defined as $E(X^n) = \sum_i x_i^n p_i = \mu_n', n \geq 1$

For continuous RV X, the n^{th} moment is defined as $E(X^n) = \int_{-\infty}^{\infty} x^n f(x)dx = \mu_n', n \geq 1$

2.Define n^{th} **Moments about Mean**

The n^{th} central moment of a discrete RV X is its moment about its mean $\overline{X}$ and is defined as

$$E\left(X - \overline{X}\right)^n = \sum_i \left(x_i - \overline{X}\right)^n p_i = \mu_n, n \geq 1$$

The n^{th} central moment of a continuous RV X is defined as

$$E\left(X - \overline{X}\right)^n = \int_{-\infty}^{\infty} \left(x - \overline{X}\right)^n f(x)dx = \mu_n, n \geq 1$$

3..Define Variance

The second moment about the mean is called variance and is represented as σ_x^2

$$\sigma_x^2 = E\left[X^2\right] - \left[E(X)\right]^2 = \mu_2' - \left(\mu_1'\right)^2$$

The positive square root σ_x of the variance is called the standard deviation.

4.Define Moment Generating Functions (M.G.F)

Moment generating function of a RV X about the origin is defined as

$$M_X(t) = E(e^{tx}) = \begin{cases} \sum_x e^{tx} P(x), if X is discrete. \\ \int_{-\infty}^{\infty} e^{tx} f(x)dx, if X is contionous. \end{cases}$$

Moment generating function of a RV X about the mean is defined as

$$M_{X-\mu}(t) = E(e^{t(x-\mu)})$$

5. Properties of MGF

1. $M_{X=a}(t) = e^{-at} M_X(t)$

Proof:

$$M_{X=a}(t) = E(e^{t(x-a)}) = E(e^{tx}.e^{-at}) = E(e^{tx})e^{-at} = e^{-at} M_X(t)$$

2.If X and Y are two independent RVs, then $M_{X+Y}(t) = M_X(t).M_Y(t)$

Proof:

$$M_{X+Y}(t) = E(e^{t(X+Y)}) = E(e^{tX+tY}) = E(e^{tX}.e^{tY}) = E(e^{tX}).E(e^{tY}) = M_X(t).M_Y(t)$$

3. $If M_X(t) = E(e^{tx}) then M_{cX}(t) = M_X(ct)$

Proof:

$$M_{cX}(t) = E(e^{tcX}) = E(e^{(ct)X}) = M_X(ct)$$

4.If Y=aX+b then $M_Y(t) = e^{bt} M_X(at)$ where $M_X(t)$=MGF of X.

5.If $M_{X_1}(t) = M_{X_2}(t)$ for all t, then $F_{X_1}(x) = F_{X_2}(x)$ for all x.

UNIT-I RANDOM VARIABLE

1. If the RV X takes the values 1, 2, 3 and 4 such that *2P(X=1)=3P(X=2)=P(X=3)=5P(X=4)*, find the probability distribution and cumulative distribution function of X.

2. A RV X has the following probability distribution.

X:	-2	-1	0	1	2	3
P(x):	0.1	K	0.2	2K	0.3	3K

Find (1) *K*, (2) *P(X<2), P(-2< X <2)*, (3) CDF of X, (4) Mean of X.

3. If X is RV with probability distribution

X:	1	2	3
P(X):	1/6	1/3	1/2

Find its mean and variance and $E(4X^3+3X+11)$.

4. A RV X has the following probability distribution.

X:	0	1	2	3	4	5	6	7
P(x):	0	K	2K	2K	3K	K^2	$2K^2$	$7K^2+K$

Find (1) *K*, (2) *P(X<2), P(1.5 <X< 4.5/X>2)*, (3) The smallest value of λ for which $P(X \le \lambda) > 1/2$.

5. A RV X has the following probability distribution.

X:	0	1	2	3	4
P(x):	K	3K	5K	7K	9K

Find (1) *K*, (2) *P(X<3)* and *P(0<X< 4)*, (3) Find the distribution function of X.

6. If the density function of a continuous RV X is given by $f(x) = \begin{cases} ax, & 0 \le x \le 1 \\ a, & 1 \le x \le 2 \\ 3a - ax, & 2 \le x \le 3 \\ 0, & Otherwise \end{cases}$

Find i)a ii)CDF of X.

7. A continuous RV X that can assume any value between x=2 and x=5 has a density function given by $f(x) = k(1+x)$. Fnd *P(X<4)*.

8. If the density function of a continuous RV X is given by $f(x) = kx^2e^{-x}, x > 0$. Find k, mean and variance.

9. If the cdf of a continuous RV X is given by $F(x) = \begin{cases} 0, & x < 0 \\ x^2, & 0 \le x < \frac{1}{2} \\ 1 - \frac{3}{25}(3 - x^2), & \frac{1}{2} \le x < 3 \\ 1, & x \ge 3 \end{cases}$

Find the pdf of X and evaluate $P(|X| \le 1)$ and $P(\frac{1}{3} \le X < 4)$.

10. A continuous RV X has the pdf $f(x) = Kx^2e^{-x}, x \geq 0$. Find the r^{th} moment about origin.Hence find mean and variance of X.

11. Find the mean, variance and moment generating function of a binomialdistribution.

12. 6 dice are thrown 729 times.How many times do you expect atleast three dice to show 5or 6?

.13. It is known that the probability of an item produced by a certain machine will be defective is 0.05. If the produced items are sent to the market in packets of 20, find the no. of packets containing atleast, exactly and atmost 2 defective items in a consignment of 1000 packets using (i) Binomial distribution

14. Find mean , variance and MGF of Geometric distribution.

15. The pdf of the length of the time that a person speaks over phone is

$$f(x) = \begin{cases} Be^{\frac{-x}{6}}, & x > 0 \\ 0, & otherwise \end{cases}$$

what is the probability that the person wil talk for (i) more than 8 minutes (ii)less than 4 minutes (iii) between 4 and 8 minutes.

16. State and prove the memory less property of the exponential distribution.

17. If the service life, in hours, of a semiconductor is a RV having a Weibull distribution with the parameters α = 0.025 and β = 0.5,
    1. How long can such a semiconductor be expected to last?
    2. What is the probability that such a semiconductor will still be in operating condition after 4000 hours?

Unit II Two Dimensional Random Variables

1.Define Two-dimensional Random variables.

Let S be the sample space associated with a random experiment E. Let X=X(S) and Y=Y(S) be two functions each assigning a real number to each $s \in S$. Then (X,Y) is called a two dimensional random variable.

2.Joint probability distribution of (X, Y)

Let (X,Y) be a two dimensional discrete random variable. Let $P(X=x_i, Y=y_j)=p_{ij}$. p_{ij} is called the probability function of (X,Y) or joint probability distribution. If the following conditions are satisfied

1.$p_{ij} \ge 0$ for all i and j

2.$\sum_j \sum_i p_{ij} = 1$

The set of triples (x_i, y_j, p_{ij}) i=1,2,3……. And j=1,2,3……is called the Joint probability distribution of (X,Y)

3.Joint probability density function

If (X,Y) is a two-dimensional continuous RV such that

$$P\left\{x - \frac{dx}{2} \le X \le x + \frac{dx}{2} and y - \frac{dy}{2} \le Y \le y + \frac{dy}{2}\right\} = f(x,y)dxdy$$

Then f(x,y) is called the joint pdf of (X,Y) provided the following conditions satisfied.

1. $f(x,y) \ge 0$ *for all* $(x,y) \in (-\infty, \infty)$
2. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1$ *and* $f(x,y) \ge 0$ *for all* $(x,y) \in (-\infty, \infty)$

4.Joint cumulative distribution function (joint cdf)

If (X,Y) is a two dimensional RV then $F(x,y) = P(X \le x, Y \le y)$ is called joint cdf of (X,Y)

In the discrete case,

$$F(x,y) = \sum_{\substack{j \\ y_j \le y}} \sum_{\substack{i \\ x_i \le x}} p_{ij}$$

In the continuous case,

$$F(x,y) = P(-\infty < X \le x, -\infty < Y \le y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f(x,y)dxdy$$

5. The following table gives the joint probability distribution of X and Y. Find the mariginal density functions of X and Y.

Y /X	**1**	**2**	**3**
1	**0.1**	**0.1**	**0.2**
2	**0.2**	**0.3**	**0.1**

Answer:

The marginal density of X

$$P(X = x_i) = p_{i*} = \sum_j p_{ij}$$

X	1	2	3
P(X)	0.3	0.4	0.3

The marginal density of Y

$$P(Y = y_j) = p_{*j} = \sum_i p_{ij}$$

Y	1	2
P(Y)	0.4	0.6

6.If $f(x,y)=kxye^{-(x^2+y^2)}$, $x\ge 0, y\ge 0$ **is the joint pdf, find** k.

Answer:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dydx=1 \Rightarrow \int_{0}^{\infty}\int_{0}^{\infty} kxye^{-(x^2+y^2)}dydx=1$$

$$k\int_0^{\infty} xe^{-x^2}dx\int_0^{\infty} ye^{-y^2}dx=1 \Rightarrow \frac{k}{4}=1$$

$$\therefore k=4$$

7. Let the joint pdf of X and Y is given by $f(x,y)=\begin{cases} cx(1-x), 0\le x\le y\le 1 \\ 0 \quad\quad ,otherwise\end{cases}$

Find the value of C.

Answer:

$$\int_0^1\int_0^y Cx(1-x)dxdy=1 \Rightarrow \frac{C}{6}\int_0^1 (3y^2-2y^3)dy=1 \Rightarrow \frac{C}{6}\left[1-\frac{1}{2}\right]=1$$

8. The joint p.m.f of (X,Y) is given by $P(x,y)=k(2x+3y),\ x=0,1,2;\ y=1,2,3.$ **.Find the marginal probability distribution of X.**

Answer:

X \Y	1	2	3
0	3k	6k	9k
1	5k	8k	11k
2	7k	10k	13k

$$\sum_y\sum_x P(x,y)=1 \Rightarrow 72k=1 \therefore k=\frac{1}{72}$$

Marginal distribution of X:

X	0	1	2
P(X)	18/72	24/72	30/72

9. If X and Y are independent RVs with variances 8and 5.find the variance of 3X+4Y.

Answer:

Given Var(X)=8 and Var(Y)=5

To find:var(3X-4Y)

We know that Var(aX - bY)=a^2Var(X)+b^2Var(Y)

var(3X- 4Y)=3^2Var(X)+4^2Var(Y) =(9)(8)+(16)(5)=152

10. Find the value of k if $f(x,y)=k(1-x)(1-y)$ **for** $0<x,y<1$ **is to be joint density function.**

Answer:

We know that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dydx=1$

$$\int_0^1\int_0^1 k(1-x)(1-y)dxdy=1 \Rightarrow k\left[\int_0^1(1-x)dx\right]\left[\int_0^1(1-y)dy\right]=1$$

$$k\left[x-\frac{x^2}{2}\right]_0^1\left[y-\frac{y^2}{2}\right]_0^1=1 \Rightarrow \frac{k}{4}=1 \qquad \therefore k=4$$

11. If X and Y are random variables having the joint p.d.f $f(x,y) = \frac{1}{8}(6-x-y),\quad 0<x<2,\ 2<y<4$**, find P(X<1,Y<3)**

Answer:

$$P(X<1, Y<3) = \frac{1}{8}\int_0^1\int_2^3 (6-x-y)\,dy\,dx = \frac{1}{8}\int_0^1\left(\frac{7}{2}-x\right)dx = \frac{3}{8}$$

12.Marginal probability distribution(Discrete case)

Let (X,Y) be a two dimensional discrete RV and $p_{ij}=P(X=x_i,Y=y_j)$ then

$$P(X=x_i) = p_{i*} = \sum_j p_{ij}$$

is called the Marginal probability function.

The collection of pairs $\{x_i,p_{i*}\}$ is called the Marginal probability distribution of X.

If $P(Y=y_j) = p_{*j} = \sum_i p_{ij}$ then the collection of pairs $\{x_i,p_{*j}\}$ is called the Marginal probability distribution of Y.

13.Marginal density function (Continuous case)

Let f(x,y) be the joint pdf of a continuous two dimensional RV(X,Y).The marginal density function of X is defined by $f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

The marginal density function of Y is defined by $f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$

14.Conditional probability function

If $p_{ij}=P(X=x_i,Y=y_j)$ is the Joint probability function of a two dimensional discrete RV(X,Y) then the conditional probability function X given $Y=y_j$ is defined by

$$P\left[\frac{X=x_i}{Y=y_j}\right] = \frac{P[X=x_i \cap Y=y_j]}{P[Y=y_j]}$$

The conditional probability function Y given $X=x_i$ is defined by

$$P\left[\frac{Y=y_j}{X=x_i}\right] = \frac{P[X=x_i \cap Y=y_j]}{P[X=x_i]}$$

15.Conditional density function

Let f(x,y) be the joint pdf of a continuous two dimensional RV(X,Y).Then the Conditional density function of X given Y=y is defined by $f(X/Y) = \frac{f_{XY}(x,y)}{f_Y(y)}$,where f(y)=marginal p.d.f of Y.

The Conditional density function of Y given X=x is defined by $f(Y/X) = \frac{f_{XY}(x,y)}{f_X(x)}$,

where f(x)=marginal p.d.f of X.

16.Define statistical properties

Two jointly distributed RVs X and Y are statistical independent of each other if and only if the joint probability density function equals the product of the two marginal probability density function

i.e., f(x,y)=f(x).f(y)

17. The joint p.d.f of (X,Y)is given by $f(x,y) = \frac{1}{4}(1+xy), |x| < 1, |y| < 1$

$= 0$ **,otherwise.**

Show that X and Y are not independent.

Answer:

Marginal p.d.f of X :

$$f(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_{-1}^{1} \frac{1}{4}(1+xy)dy = \frac{1}{2}, \quad -1 < x < 1$$

$$f(x) = \begin{cases} \frac{1}{2}, & -1 < x < 1 \\ 0, & otherwise \end{cases}$$

Marginal p.d.f of Y :

$$f(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_{-1}^{1} \frac{1}{4}(1+xy)dx = \frac{1}{2}, \quad -1 < y < 1$$

$$f(y) = \begin{cases} \frac{1}{2}, & -1 < y < 1 \\ 0, & otherwise \end{cases}$$

Since $f(x)f(y) \neq f(x,y)$,X and Y are not independent.

18.The conditional p.d.f of X and Y=y is given by $f\left(\frac{x}{y}\right) = \frac{x+y}{1+y}e^{-x}, 0 < x < \infty, 0 < y < \infty,$

find $P[X < 1/Y = 2]$**.**

Answer:

When y =2, $f(x/y=2) = \frac{x+2}{3}e^{-x}$

$$\therefore P[X < 1/Y = 2] = \int_0^1 \frac{x+2}{3}e^{-x}dx = \frac{1}{3}\int_0^1 xe^{-x}dx + \frac{2}{3}\int_0^1 e^{-x}dx = 1 - \frac{4}{3}e^{-1}$$

19. **The joint p.d.f of two random variables X and Y is given by**

$f(x,y) = \frac{1}{8}x(x-y), 0 < x < 2, -x < y < x$ **and** $= 0$ **, otherwise.**

Find $f(y/x)$

Answer:

$$f(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_{-x}^{x} \frac{1}{8}x(x-y)dy = \frac{x^3}{4}, \quad 0 < x < 2$$

$$f(y/x) = \frac{f(x,y)}{f(x)} = \frac{x-y}{2x^2}, -x < y < x$$

20. If the joint pdf of (X,Y) is $f(x,y) = \frac{1}{4}, \ 0 \le x, \ y < 2$ **,find** $P[X + Y \le 1]$

Answer:

$$P[X+Y \le 1] = \int_0^1 \int_0^{1-y} \frac{1}{4}dxdy = \frac{1}{4}\int_0^1 (1-y)dy = \frac{1}{8}.$$

21. If the joint pdf of (X,Y) is $f(x,y)=6e^{-2x-3y},\ x\geq 0, y\geq 0,$ find the conditional density of Y given X.

Answer:

Given $f(x,y)=6e^{-2x-3y},\ x\geq 0, y\geq 0,$

The Marginal p.d.f of X:

$$f(x)=\int_0^\infty 6e^{-2x-3y}dy=2e^{-2x},\ x\geq 0$$

Conditional density of Y given X:

$$f(y/x)=\frac{f(x,y)}{f(x)}=\frac{6e^{-2x-3y}}{2e^{-2x}}=3e^{-3y}, y\geq 0.$$

22.**Find the probability distribution of (X+Y) given the bivariate distribution of (X,Y).**

X \Y	**1**	**2**
1	**0.1**	**0.2**
2	**0.3**	**0.4**

Answer:

X+Y	P(X+Y)
2	P(2)=P(X=1,Y=1)=0.1
3	P(3)=P(X=1,Y=2)+P(X=2,Y=1)=0.2+0.3=0.5
4	P(4)=P(X=2,Y=2)=0.4

$\therefore$

X+Y	2	3	4
Probability	0.1	0.5	0.4

23. The joint p.d.f of (X,Y) is given by $f(x,y)=6e^{-(x+y)},\ 0\leq x,y\leq\infty.$ **Are X and Y independent?**

Answer:

Marginal density of X:

$$f(x)=\int_{-\infty}^{\infty} f(x,y)dy=\int_0^\infty 6e^{-(x+y)}dy=e^{-x},\ 0\leq x$$

Marginal density of Y;

$$f(y)=\int_{-\infty}^{\infty} f(x,y)dx=\int_0^\infty 6e^{-(x+y)}dx=e^{-y},\ y\leq\infty$$

$$\Rightarrow f(x)f(y)=f(x,y)$$

$\therefore$ X and Y are independent.

24.The joint p.d.f of (X,Y) is given by $f(x,y)=e^{-(x+y)}\ \ 0\leq x,y\leq\infty$ **.Are X and Y are independent?**

Answer:

Marginal densities:

f(x)= $\int_0^\infty e^{-(x+y)}dy=e^{-x}$ and f(y)= $\int_0^\infty e^{-(x+y)}dx=e^{-y}$

X and Y are independent since f(x,y)=f(x).f(y)

25. The joint p.d.f of a bivariate R.V (X,Y) is given by

$$f(x,y)=\begin{cases}4xy, & 0<x<1, y<1\infty \\ 0 & ,otherwise\end{cases}$$ **.Find p(X+Y<1)**

Answer:

$$P[X+Y<1]=\int_0^1\int_0^{1-y}4xydxdy=2\int_0^1 y(1-y)^2dy$$

$$=2\left[\frac{y^2}{2}-\frac{2y^3}{3}+\frac{y^4}{4}\right]_0^1$$

$$=2\left[\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right]=\frac{1}{6}$$

26. Define Co – Variance:

If X and Y are two two r.v.s then co – variance between them is defined as

Cov (X, Y) = E {X – E (X)} {Y – E (Y)}

(ie) Cov (X, Y) = E (XY) – E(X) E (Y)

27. State the properties of Co – variance;

1. If X and Y are two independent variables, then Cov (X,Y) = 0. But the Converse need not be true
2. Cov (aX, bY) = ab Cov (X,Y)
3. Cov (X + a,Y +b) = Cov (X,Y)
4. $Cov\left(\frac{X-\overline{X}}{\sigma_X},\frac{Y-\overline{Y}}{\sigma_Y}\right)=\frac{1}{\sigma_X\sigma_Y}Cov(X,Y)$
5. Cov(aX+b,cY+d)=acCov(X,Y)
6. Cov (X+Y,Z) = Cov (X,Z) + Cov (Y,Z)
7. $Cov(aX+bY,cX+dY)=ac\sigma_X{}^2+bd\sigma_Y{}^2+(ad+bc)Cov(X,Y)$

$where\,\sigma_X{}^2=Cov(X,X)=\text{var}(X)\,and\,\sigma_Y{}^2=Cov(Y,Y)=\text{var}(Y)$

28.Show that Cov(aX+b,cY+d)=acCov(X,Y)

Answer:

Take U= aX+b and V= cY+d

Then E(U)=aE(X)+b and E(V)= cE(Y)+d

U-E(U)= a[X-E(X)] and V-E(V)=c[Y-E(Y)]

Cov(aX+b,cY+d)= Cov(U,V)=E[{U-E(U)}{V-E(V)}] = E[a{X-E(X)} c{Y-E(Y)}]

=ac E[{X-E(X)}{Y-E(Y)}]=acCov(X,Y)

29.If X&Y are independent R.V's ,what are the values of Var(X_1+X_2)and Var(X_1-X_2)

Answer:

$\text{Var}(X_1\pm X_2)=\text{Var}(X_1)+\text{Var}(X_2)$ (Since X andY are independent RV then

$\text{Var}(aX\pm bX)=a^2\text{Var}(X)+b^2\text{Var}(X)$)

30. If Y_1& Y_2 are independent R.V's ,then covariance (Y_1,Y_2)=0.Is the converse of the above statement true?Justify your answer.

Answer:

The converse is not true . Consider

$X - N(0,1)\text{and } Y = X^2 \quad \sin ceX - N(0,1),$

$E(X)=0;\ E(X^3)=E(XY)=0\sin ce\ \ all\ odd\, moments\quad vanish.$

$\therefore \text{cov}(XY)=E(XY)-E(X)E(Y)=E(X^3)-E(X)E(Y)=0$

$\therefore \text{cov}(XY)=0\quad but\quad X\,\&\,Y\quad are\, independent$

31. Show that $\operatorname{cov}^2(X,Y) \le \operatorname{var}(X)\operatorname{var}(Y)$

Answer:

$$\operatorname{cov}(X,Y) = E(XY) - E(X)E(Y)$$

We know that $[E(XY)]^2 \le E(X^2)E(Y^2)$

$$\operatorname{cov}^2(X,Y) = [E(XY)]^2 + [E(X)]^2[E(Y)]^2 - 2E(XY)E(X)E(Y)$$
$$\le E(X)^2E(Y)^2 + [E(X)]^2[E(Y)]^2 - 2E(XY)E(X)E(Y)$$
$$\le E(X)^2E(Y)^2 + [E(X)]^2[E(Y)]^2 - E(X^2)E(Y)^2 - E(Y^2)E(X)^2$$
$$= \{E(X^2) - [E(X)]^2\}\{E(Y^2) - [E(Y)]^2\} \le \operatorname{var}(X)\operatorname{var}(Y)$$

32.If X and Y are independent random variable find covariance between X+Y and X-Y.

Answer:

$$\operatorname{cov}[X+Y, X-Y] = E[(X+Y)(X-Y)] - [E(X+Y)E(X-Y)]$$
$$= E[X^2] - E[Y^2] - [E(X)]^2 + [E(Y)]^2$$
$$= \operatorname{var}(X) - \operatorname{var}(Y)$$

33. X and Y are independent random variables with variances 2 and 3.Find the variance 3X+4Y.

Answer:

Given var(X) = 2, var(Y) = 3

We know that var(aX+Y) = a^2var(X) + var(Y)

And var(aX+bY) = a^2var(X) + b^2var(Y)

var(3X+4Y) = 3^2var(X) + 4^2var(Y) = 9(2) + 16(3) = 66

34. Define correlation

The correlation between two RVs X and Y is defined as

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(xy)\,dxdy$$

35. Define uncorrelated

Two RVs are uncorrelated with each other, if the correlation between X and Y is equal to the product of their means. i.e.,E[XY]=E[X].E[Y]

36. If the joint pdf of (X,Y) is given by $f(x,y) = e^{-(x+y)}, x \ge 0, y \ge 0.$ **find E(XY).**

Answer:

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)dxdy = \int_0^{\infty}\int_0^{\infty} xye^{-(x+y)}dxdy = \int_0^{\infty} xe^{-x}dx\int_0^{\infty} ye^{-y}dy = 1$$

37. A R.V X is uniformly distributed over(-1,1) and Y=X^2. Check if X and Y are correlated?

Answer:

Given X is uniformly distributed in (-1,1),pdf of X is $f(x) = \frac{1}{b-a} = \frac{1}{2},\ -1 \le x \le 1$

$$E(X) = \frac{1}{2}\int_{-1}^{1} xdx = 0 \text{ and } E(XY) = E(X^3) = 0$$

$$\therefore \operatorname{cov}(X,Y) = E(XY) - E(X)E(Y) = 0 \qquad \Rightarrow r(X,Y) = 0$$

Hence X and Y are uncorrelated.

38. X and Y are discrete random variables. If $\text{var}(X) = \text{var}(Y) = \sigma^2$,

$\text{cov}(X,Y) = \frac{\sigma^2}{2}$, **find** $\text{var}(2X - 3Y)$

Answer:

$$\text{var}(2X - 3Y) = 4\,\text{var}(X) + 9\,\text{var}(Y) - 12\,\text{cov}(X,Y)$$

$$= 13\sigma^2 - 12\frac{\sigma^2}{2} = 7\sigma^2$$

39. If $\text{var}(X) = \text{var}(Y) = \sigma^2$, $\text{cov}(X,Y) = \frac{\sigma^2}{2}$, **find the correlation between** $2X + 3$ **and** $2Y - 3$

Answer:

$$r(aX + b, cY + d) = \frac{ac}{|ac|} r(X,Y) \ \ where\ a \neq 0, c \neq 0$$

$$\therefore r(2X + 3, 2Y - 3) = \frac{4}{|4|} r(X,Y) = r(X,Y) = \frac{\text{cov}(X,Y)}{\sigma_X \sigma_Y} = \frac{\sigma^2/2}{\sigma.\sigma} = \frac{1}{2}$$

40. Two independent random variables X and Y have 36 and 16.Find the correlation co-efficient between X+Y and X-Y

Answer:

$$\therefore r(X + Y, X - Y) = \frac{\sigma_X^2 - \sigma_Y^2}{\sigma_X^2 + \sigma_Y^2} = \frac{36 - 16}{36 + 16} = \frac{20}{52} = \frac{4}{13}$$

41. If the lines of regression of Y on X and X on Y are respectively $a_1X + b_1Y + c_1 = 0$ **and** $a_2X + b_2Y + c_2 = 0$ **,prove that** $a_1b_2 \leq a_2b_1$**.**

Answer:

$$b_{yx} = -\frac{a_1}{b_1} \quad and \quad b_{xy} = -\frac{b_2}{a_2}$$

Since $r^2 = b_{yx}b_{yx} \leq 1 \quad \Rightarrow \frac{a_1}{b_1}.\frac{b_2}{a_2} \leq 1$

$$\therefore a_1b_2 \leq a_2b_1$$

42. State the equations of the two regression lines. what is the angle between them?

Answer:

Regression lines:

$$y - \bar{y} = r\frac{\sigma y}{\sigma x}(x - \bar{x}) \ \ and\ \ x - \bar{x} = r\frac{\sigma x}{\sigma y}(y - \bar{y})$$

$$\text{Angle } \theta = \tan^{-1}\left[\frac{1 - r^2}{r}\left(\frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}\right)\right]$$

43. The regression lines between two random variables X and Y is given by $3X + Y = 10$ **and** $3X + 4Y = 12$ **.Find the correlation between X and Y.**

Answer:

$$3X + 4Y = 12 \qquad \Rightarrow b_{yx} = -\frac{3}{4}$$

$$3X + Y = 10 \qquad \Rightarrow b_{xy} = -\frac{1}{3}$$

$$r^2 = \left(-\frac{3}{4}\right)\left(-\frac{1}{3}\right) = \frac{1}{4} \qquad \Rightarrow r = -\frac{1}{2}$$

44. Distinguish between correlation and regression.

Answer:

By correlation we mean the casual relationship between two or more variables.
By regression we mean the average relationship between two or more variables.

45. State the Central Limit Theorem.

Answer:

If $x_1, x_2, \ldots\ldots\ldots, x_n$ are n independent identically distributed RVs with men μ and S.D σ and if $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, then the variate $z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}}$ has a distribution that approaches the standard normal distribution as $n \to \infty$ provided the m.g.f of x_i exist.

46. The lifetime of a certain kind of electric bulb may be considered as a RV with mean 1200 hours and S.D 250 hours. Find the probability that the average life time of exceeds 1250 hours using central limit theorem.

Solution:

Let X denote the life time of the 60 bulbs.
Then μ = E(X)= 1200 hrs. and Var(X)=(S.D)^2= σ^2=(250)^{2}hrs.
Let $\overline{X}$ denote the average life time of 60 bulbs.

By Central Limit Theorem, $\overline{X}$ follows $N\left(\mu, \frac{\sigma^2}{n}\right)$.

Let $Z = \frac{\overline{X} - \mu}{\sigma / \sqrt{n}}$ be the standard normal variable

$$P[\overline{X} > 1250] = P[Z > 1.55]$$
$$= 0.5 - P[0 < Z < 1.55]$$
$$= 0.5 - 0.4394 = 0.0606$$

PART-B

UNIT II – TWO DIMENSIONAL RANDOM VARIABLES

1. The joint probability mass function of (X,Y) is given by p(x,y) = k(2x+3y), x=0,1,2;y=1,2,3.

Find (i) the Marginal distributions of X and y.
(ii) Conditional distributions P[X/Y= y] and P[Y/X= x].
Also find the probability distribution of (X + Y).

2. If the joint density of two random variables X and Y is given by $f(x,y) = x + y, 0 \le x \le 1, 0 \le y \le 1$ and =0,elsewhere. Obtain the marginal and conditional distributions.Also find cov(X,Y)

3. If the joint density function of the RVs X and Y is given by $f(x,y) = k(6 - x - y), 0 < x < 2, 2 < y < 4$, find i) k, ii) *P(X<1,Y<3)*,iii) *P(X+Y<3)* iv)*P(X<1/Y<3)*

4. If X and Y are RVs having the joint pdf f(x,y)=(1/8)(6-x-y), 0<x<2, 2<y<4.Find *P(X<1∩Y<3), P(X+Y< 3) and P(X<1,Y<3).*

5. Two dimensional R.V (X,Y) have the joint pdf $f(x,y) = 8xy, 0 < x < 1$ and $= 0, elsewhere$

(i) find $P\left[X < \frac{1}{2} \cap Y < \frac{1}{4}\right]$

(ii) Find the Marginal and Conditional distributions.

(iii) Are X and Y are independent ?

6. Compute the coefficient of correlation between X and Y, using the following data:

X	65	67	66	71	67	70	68	69
Y	67	68	68	70	64	67	72	70

7. Two random variables X and Y have the joint probability density function
$$f(x,y) = \begin{cases} k(4 - x - y) \\ 0 \end{cases} \quad 0 \le x \le 2,\ 0 \le y \le 2$$. Find k and Cov(X,Y)and Correlation coefficient between X and Y.

8. If X and Y are RVs having the joint pdf $f(x,y) = xy^2 + \frac{x^2}{8} 0 \le x \le 2,\ 0 \le y \le 1$.Find
$P\left[X > 1 / Y < \frac{1}{2}\right] P\left[Y < \frac{1}{2} / X > 1\right]$ and

9. In a paritially destroyed laboratory record of an analysis of correlation data,the following results only are legible.
Variance of x=9. Regression equations: 8x-10y + 66 = 0 ;40x-18y=214.
Find (i) the mean values of x and y.
(ii) the correlation coefficient between X and Y
(iii) the standard deviation of y.

10. A distribution with unknown mean μ has the variance equal to 1.5. Use central limit theorem, to find how large a sample should be taken from the distribution in order that the probability will be atleast 0.95 that the sample mean will be within 0.5 of the population mean.

Unit III
Classification of Random Process

1. **Define Random processes and give an example of a random process.**

A Random process is a collection of R.V $\{X(s,t)\}$ that are functions of a real variable namely time t where $s \in S$ and $t \in T$

Example:

$X(t) = A\cos(\omega t + \theta)$ where θ is uniformly distributed in $(0, 2\pi)$ where A and ω are constants.

2. State the four classifications of Random processes.

The Random processes is classified into four types

(i)Discrete random sequence

If both T and S are discrete then Random processes is called a discrete Random sequence.

(ii)Discrete random processes

If T is continuous and S is discrete then Random processes is called a Discrete Random processes.

(iii)Continuous random sequence

If T is discrete and S is continuous then Random processes is called a Continuous Random sequence.

(iv)Continuous random processes

If T &S are continuous then Random processes is called a continuous Random processes.

3**. Define stationary Random processes.**

If certain probability distributions or averages do not depend on t, then the random process $\{X(t)\}$ is called stationary.

4.**Define first order stationary Random processes.**

A random processes $\{X(t)\}$ is said to be a first order SSS process if $f(x_1, t_1 + \delta) = f(x_1, t_1)$ (i.e.) the first order density of a stationary process $\{X(t)\}$ is independent of time t

5.**Define second order stationary Random processes**

A RP $\{X(t)\}$ is said to be second order SSS if $f(x_1, x_2, t_1, t_2) = f(x_1, x_2, t_1 + h, t_2 + h)$ where $f(x_1, x_2, t_1, t_2)$ is the joint PDF of $\{X(t_1), X(t_2)\}$.

6.**Define strict sense stationary Random processes**

Sol: A RP $\{X(t)\}$ is called a SSS process if the joint distribution $X(t_1)X(t_{21})X(t_3).........X(t_n)$ is the same as that of $X(t_1 + h)X(t_2 + h)X(t_3 + h).........X(t_n + h)$ for all $t_1, t_2, t_3.........t_n$ and h >0 and for $n \geq 1$.

7.**Define wide sense stationary Random processes**

A RP $\{X(t)\}$ is called WSS if $E\{X(t)\}$ is constant and $E[X(t)X(t+\tau)] = R_{xx}(\tau)$ (i.e.) ACF is a function of τ only.

8.**Define jointly strict sense stationary Random processes**

Sol: Two real valued Random Processes $\{X(t)\}$ and $\{Y(t)\}$ are said to be jointly stationary in the strict sense if the joint distribution of the $\{X(t)\}$ and $\{Y(t)\}$ are invariant under translation of time.

9.**Define jointly wide sense stationary Random processes**

Sol: Two real valued Random Processes $\{X(t)\}$ and $\{Y(t)\}$ are said to be jointly stationary in the wide sense if each process is individually a WSS process and $R_{XY}(t_1, t_2)$ is a function of t_1, t_2 only.

10. **Define Evolutionary Random processes and give an example**.

Sol: A Random processes that is not stationary in any sense is called an Evolutionary process. Example: Poisson process.

11.**If $\{X(t)\}$ is a WSS with auto correlation $R(\tau) = Ae^{-\alpha|\tau|}$, determine the second order moment of the random variable X (8) – X (5).**

Sol: Given $R_{xx}(\tau) = Ae^{-\alpha|\tau|}$ (i.e.) $R_{xx}(t_1, t_2) = Ae^{-\alpha|t_1 - t_2|}$

(i.e.) $E(X(t_1).X(t_2)) = Ae^{-\alpha|t_1 - t_2|}$ .... (1)

$\therefore E(X^2(t)) = E(X(t)X(t)) = R_{xx}(t,t) = Ae^{-\alpha(0)} = A$

$\therefore E(X^2(8)) = A \,\&\, E(X^2(5)) = A \;\; \therefore E(X(8)X(5)) = R_{xx}(8,5) = A.e^{-3\alpha}$.

Now second order moment of $\{X(8) - X(5)\}$ is given by

$$E(X(8) - X(5))^2 = E(X^2(8) + X^2(5) - 2X(8)X(5))$$

$$= E(X^2(8)) + E(X^2(5)) - 2E(X(8)X(5))$$

$$= A + A - 2Ae^{-3\alpha} = 2A(1 - e^{-3\alpha})$$

12.**Verify whether the sine wave process $\{X(t)\}$, where $X(t) = Y\cos\omega t$ where Y is uniformly distributed in (0,1) is a SSS process.**

Sol: $F(x) = P(X(t) \le x) = P(Y\cos\omega t \le x)$

$$= \begin{cases} P\left(Y \le \dfrac{x}{\cos\omega t}\right) if \cos\omega t > 0 \\ P\left(Y \ge \dfrac{x}{\cos\omega t}\right) if \cos\omega t < 0 \end{cases}$$

$$F_X(x) = \begin{cases} F_Y\left(\dfrac{x}{\cos\omega t}\right) if \cos\omega t > 0 \\ 1 - F_Y\left(\dfrac{x}{\cos\omega t}\right) if \cos\omega t < 0 \end{cases}$$

$$\therefore f_{X(t)}(x) = \frac{1}{|\cos\omega t|} f_Y\left(\frac{x}{\cos\omega t}\right) = \text{a function of t}$$

If {X(t)} is to be a SSS process, its first order density must be independent of t. Therefore, {X(t)}is not a SSS process.

13.**Consider a random variable $Z(t) = X_1\cos\omega_0 t - X_2\sin\omega_0 t$ where X_1 and X_2 are independent Gaussian random variables with zero mean and variance σ^2 find E(Z) and E(Z^2)**

Sol: Given $E(X_1) = 0 = E(X_2) \,\&\, Var(X_1) = \sigma^2 = Var(X_2)$

$\Rightarrow E(X_1^{\,2}) = \sigma^2 = E(X_2^{\,2})$

$$E(Z) = E(X_1 \cos\omega_0 t - X_2 \sin\omega_0 t) = 0$$

$$E(Z^2) = E(X_1 \cos\omega_0 t - X_2 \sin\omega_0 t)^2$$

$$= E(X_1^{\,2})\cos^2\omega_0 t + E(X_2^{\,2})\sin^2\omega_0 t - E(X_1 X_2)\cos\omega_0 t \sin\omega_0 t$$

$$= \sigma^2(\cos^2\omega t + \sin^2\omega t) - E(X_1)E(X_2)\cos\omega_0 t\sin\omega_0 t \qquad \because X_1 \& X_2 \text{ are independent}$$

$$= \sigma^2 - 0 = \sigma^2.$$

14. Consider the random process $X(t) = \cos(\omega_0 t + \theta)$ where θ is uniformly distributed in $(-\pi, \pi)$. Check whether X(t) is stationary or not?

Answer:

$$E[X(t)] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(\omega_0 t + \theta)d\theta = \frac{1}{2\pi}[\sin(\omega_0 t + \pi) - \sin(\omega_0 t - \pi)] = \frac{1}{2\pi}[-\sin(\omega_0 t) + \sin(\omega_0 t)] = 0$$

$$E[X^2(t)] = \frac{1}{4\pi}\left[\frac{\theta - 2\sin(\omega_0 t + \theta)}{2}\right]_{-\pi}^{\pi} = \frac{1}{2}$$

15.**When is a random process said to be ergodic? Give an example**

Answer: A R.P {X(t)} is ergodic if its ensembled averages equal to appropriate time averages. Example: $X(t) = A\cos(\omega t + \theta)$ where θ is uniformly distributed in $(0, 2\pi)$ is mean ergodic.

16. **Define Markov Process.**

Sol: If for $t_1 < t_2 < t_3 < t_4 \ldots\ldots\ldots\ldots < t_n < t$ then

$$P(X(t) \le x / X(t_1) = x_1, X(t_2) = x_2, \ldots\ldots\ldots X(t_n) = x_n) = P(X(t) \le x / X(t_n) = x_n)$$

Then the process $\{X(t)\}$ is called a Markov process.

17.**Define Markov chain.**

Sol: A Discrete parameter Markov process is called Markov chain.

18.**Define one step transition probability.**

Sol: The one step probability $P\lfloor X_n = a_j / X_{n-1} = a_i \rfloor$ is called the one step probability from the state a_i to a_j at the n^{th} step and is denoted by $P_{ij}(n-1, n)$

19.**The one step tpm of a Markov chain with states 0 and 1 is given as $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Draw the Transition diagram. Is it Irreducible Markov chain?**

Sol: Yes it is irreducible since each state can be reached from any other state.

20.**Prove that the matrix P = $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$ is the tpm of an irreducible Markov chain.**

Sol: $P^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$ $P^3 = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$

Here $P_{11}^{(3)} > 0. P_{13}^{(2)} > 0, P_{21}^{(2)} > 0, P_{22}^{(2)} > 0, P_{33}^{(2)} > 0$ and for all other $P_{ij}^{(1)} > 0$

Therefore the chain is irreducible.

21. **State the postulates of a Poisson process.**

Let $\{X(t)\}$ = number of times an event A say, occurred up to time 't' so that the sequence $\{X(t)\}$, $t \geq 0$ forms a Poisson process with parameter λ.

(i) P[1 occurrence in $(t, t+\Delta t)$]= $\lambda \Delta t$

(ii) P[0 occurrence in $(t, t+\Delta t)$]=1- $\lambda \Delta t$

(iii) P[2 or more occurrence in $(t, t+\Delta t)$]=0

(iv) X(t) is independent of the number of occurrences of the event in any interval prior and after the interval (0,t).

(v) The probability that the event occurs a specified number of times in (t_0,t_0+t) depends only on t, but not on t_0.

22.**State any two properties of Poisson process**

Sol: (i) The Poisson process is a Markov process

(ii) Sum of two independent Poisson processes is a Poisson process

(iii) The difference of two independent Poisson processes is not a Poisson process.

23.**If the customers arrived at a counter in accordance with a Poisson process with a mean rate of 2 per minute, find the probability that the interval between two consecutive arrivals is more than one minute**.

Sol: The interval T between 2 consecutive arrivals follows an exponential distribution with

parameter $\lambda = 2, P(T > 1) = \int_1^\infty 2e^{-2t} dt = e^{-2} = 0.135.$

24.**A bank receives on an average** $\lambda = 6$ **bad checks per day, what are the probabilities that it will receive (i) 4 bad checks on any given day (ii) 10 bad checks over any 2 consecutive days.**

Sol: $P(X(t) = n) = \dfrac{e^{-\lambda t}.(\lambda t)^n}{n!} = \dfrac{e^{-6t}(6t)^n}{n!}, n = 0,1,2....$

(i) $P(X(1) = 4) = \dfrac{e^{-6}(6)^4}{4!} = 0.1338$

(ii) $P(X(2) = 10) = \dfrac{e^{-12}(12)^{10}}{10!} = 0.1048$

25.**Suppose the customers arrive at a bank according to a Poisson process with a mean rate of 3 per minute. Find the probability that during a time interval of 2 minutes exactly 4 customers arrive**

Sol: (i) $P(X(t) = n) = \dfrac{e^{-3t}(3t)^n}{n!}, n = 0,1,2.....$

(ii) $P(X(2) = 4) = \dfrac{e^{-6}(6)^4}{4!} = 0.1338.$

26.**Consider a Markov chain with two states and transition probability matrix**

$P = \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix}$**.Find the stationary probabilities of the chain.**

Sol: $(\pi_1,\pi_2)\begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix} = (\pi_1,\pi_2) \qquad \pi_1+\pi_2 = 1$

$$\frac{3}{4}\pi_1 + \frac{\pi_2}{4} = \pi_1 \Rightarrow \frac{\pi_1}{4} - \frac{\pi_2}{2} = 0. \qquad \therefore \pi_1 = 2\pi_2$$

$$\therefore \pi_1 = \frac{2}{3}, \pi_2 = \frac{1}{3}.$$

27.**Customers arrive a large store randomly at an average rate of 240 per hour. What is the probability that during a two-minute interval no one will arrive.**

Sol: $P(X(t)=n) = \dfrac{e^{-4t}.(4t)^n}{n!}, n = 0,1,2....$ since $\lambda = \dfrac{240}{60} = 4$

$\therefore P(X(2)=0) = e^{-8} = 0.0003.$

28.**The no of arrivals at the reginal computer centre at express service counter between 12 noon and 3 p.m has a Poison distribution with a mean of 1.2 per minute. Find the probability of no arrivals during a given 1-minute interval.**

Sol: $P(X(t)=n) = \dfrac{e^{-1.2t}.(1.2t)^n}{n!}, n = 0,1,2....$

$P(X(1)=0) = e^{-1.2} = 0.3012.$

29.**Define Gaussian or Normal process.**

Sol: A real valued RP $\{X(t)\}$ is called a Gaussian process or normal process if the random variables $X(t_1), X(t_2),.....X(t_n)$ are jointly normal for any n and for any set t_i's.

30. State the properties of a Gaussian process.

Sol: (i) If a Gaussian process is wide sense stationary, it is also a strict sense stationary.

(ii) If the member functions of a Gaussian process are uncorrelated, then they are independent.

(iii)If the input $\{X(t)\}$ of a linear system is a Gaussian process, the output will also be a Gaussian process.

31.**Define Square law Detector Process**

Sol: If $\{X(t)\}$ is a zero mean stationary Gaussian process and $Y(t) = X^2(t)$,then $\{Y(t)\}$ is a Square law Detector Process.

This process $Y(t) = X^2(t)$ is wide sense stationary.

32.**Define Full wave Linear Detector process.**

Sol: If $\{X(t)\}$ is a zero mean stationary Gaussian process and $Y(t) = |X(t)|$, then $\{Y(t)\}$ is called a Full wave Linear Detector process.

For this process $\{Y(t)\}, E[Y(t)] = \sqrt{\dfrac{2}{\pi}} R_{xx}(0)$ and

$R_{yy}(\tau) = \dfrac{2}{\pi} R_{xx}(0)[\cos\alpha + \alpha\sin\alpha]$ where $\sin\alpha = \dfrac{R_{xx}(\tau)}{R_{xx}(0)}.$

Hence $\{Y(t)\}$ is wide sense stationary.

33. **Define Half wave Linear Detector process**.

Sol: If $\{X(t)\}$ is a zero mean stationary Gaussian process and if

$Z(t) = \quad X(t)$ for $X(t) \ge 0$

0, for $X(t) < 0$.

Then Z(t) is called a Half wave Linear Detector process.

34. Define Hard Limiter process.

Sol: If $\{X(t)\}$ is a zero mean stationary Gaussian process and if

$$Y(t) = \begin{cases} +1 & \text{for } X(t) \geq 0 \\ -1, & \text{for } X(t) < 0. \end{cases}$$

Then $Y(t)$ is called a Hard limiter process or ideal limiter process.

35. For the sine wave process $X(t) = Y\cos\omega t, -\infty < t < \infty$ where ω = constant, the amplitude Y is a random variable with uniform distribution in the interval 0 and 1.check wheather the process is stationary or not.

Sol: $f(y) = \begin{cases} 1 & 0<y<1 \\ 0 & \text{Otherwise} \end{cases}$

$$E(X(t)) = \int_0^1 1.Y\cos\omega t = \cos\omega t\int_0^1 y dy = \cos\omega t. \text{ (a function of t)}$$

Therefore it is not stationary.

36. **Derive the Auto Correlation of Poisson Process**.

Sol: $R_{xx}(t_1,t_2) = E[X(t_1)X(t_2)]$

$$R_{xx}(t_1,t_2) = E[X(t_1)\{X(t_2) - X(t_1) + X(t_1)\}]$$
$$= E[X(t_1)\{X(t_2) - X(t_1)\}] + E[X^2(t_1)]$$
$$= E[X(t_1)]E[X(t_2) - X(t_1)] + E[X^2(t_1)]$$

Since $X(t)$ is a Poisson process, a process of independent increments.

$$\therefore R_{xx}(t_1,t_2) = \lambda t_1(\lambda t_2 - \lambda t_1) + \lambda_1 t_1 + \lambda_1^2 t_1^2 \text{ if } t_2 \geq t_1$$
$$\Rightarrow R_{xx}(t_1,t_2) = \lambda^2 t_1 t_2 + \lambda t_1 \text{ if } t_2 \geq t_1$$
$$\text{(or)} \Rightarrow R_{xx}(t_1,t_2) = \lambda^2 t_1 t_2 + \lambda \min\{t_1,t_2\}$$

37. Derive the Auto Covariance of Poisson process

Sol: $C(t_1,t_2) = R(t_1,t_2) - E[X(t_1)]E[X(t_2)]$

$$= \lambda^2 t_1 t_2 + \lambda t_1 - \lambda^2 t_1 t_2 = \lambda t_1 \text{ if } t_2 \geq t_1$$
$$\therefore C(t_1,t_2) = \lambda \min\{t_1,t_2\}$$

38. Define Time averages of Random process.

Sol: The time averaged mean of a sample function $X(t)$ of a random process $\{X(t)\}$ is defined as $\overline{X_T} = \lim_{t\to\infty}\frac{1}{2T}\int_{-T}^{T} X(t)dt$

The time averaged auto correlation of the Random process $\{X(t)\}$ is defined by

$$\overline{Z_T} = \lim_{t\to\infty}\frac{1}{2T}\int_{-T}^{T} X(t)X(t+\tau)dt.$$

39. If {X(t)} is a Gaussian process with $\mu(t) = 10$ *and* $C(t_1,t_2) = 16e^{-|t_1-t_2|}$,Find P{X(10)$\leq$8}.

Answer: $\mu[X(10)] = 10$ *and* $Var[X(10)] = C(10,10) = 16$

$$P[X(10) \leq 8] = P\left[\frac{X(10)-10}{4} \leq -0.5\right] = P[Z \leq -0.5] = 0.5 - P[Z \leq 0.5] = 0.5 - 0.1915 = 0.3085$$

40. If {X(t)} is a Gaussian process with $\mu(t) = 10$ *and* $C(t_1, t_2) = 16e^{-|t_1 - t_2|}$,Find the mean and variance of X(10)-X(6).

Answer:

X(10)-X(6) is also a normal R.V with mean $\mu(10) - \mu(6) = 0$ *and*

$$Var[X(10) - X(6)] = \text{var}\{X(10)\} + \text{var}\{X(6)\} - 2\,\text{cov}\{X(10), X(6)\}$$

$$= C(10,10) + C(6,6) - 2C(10,6) = 16 + 16 - 2 \times 16e^{-4} = 31.4139$$

Unit IV Classification and Spectral Densities

1. Define the ACF.

Answer:

Let $X(t_1)$ and $X(t_2)$ be two random variables. The autocorrelation of the random process{X(t)} is

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)].$$

If $t_1 = t_2 = t$, $R_{XX}(t,t) = E[X^2(t)]$ is called as mean square value of the random process.

2. State any four properties of Autocorrelation function.

Answer:

1. $R_{XX}(-\tau) = R_{XX}(\tau)$
2. $|R(\tau)| \le R(0)$
3. $R(\tau)$ *is continuous for all* τ.
4. If $R(\tau)$ is ACF of a stationary RP {X(t)} with no periodic components, then $\mu_X{}^2 = \lim_{\tau \to \infty} R(\tau)$.

3. Define the cross – correlation function.

Answer:

Let {X (t)} and {Y(t)} be two random processes. The cross-correlation is

$$R_{XY}(\tau) = E[X(t)Y(t-\tau)] .$$

4. State any two properties of cross-correlation function.

Answer:

1. $R_{YX}(-\tau) = R_{XY}(\tau)$
2. $|R_{XY}(\tau)| \le \sqrt{R_{XX}(0)R_{YY}(0)} \le \frac{1}{2}[R_{XX}(0) + R_{YY}(0)]$

5. Given the ACF for a stationary process with no periodic component is $R_{XX}(\tau) = 25 + \frac{4}{1+6\tau^2}$ .find the mean and variance of the process{X(t)}

Answer:

By the property of ACF

$$\mu_x{}^2 = \lim_{\tau \to \infty} R_{XX}(\tau) = \lim_{\tau \to \infty} 25 + \frac{4}{1+6\tau^2} = 25$$

$$\mu_x = 5$$

$E\{X^2(t)\}=R_{xx}(0)=25+4=29$

$Var\{X(t)\}=E\{X^2(t)\}-E^2\{X(t)\}=29-25=4.$

6. ACF: $R_{XX}(\tau) = \frac{25\tau^2 + 36}{6.25\tau^2 + 4}$ .find mean and variance.

7. ACF: $R_{XX}(\tau) = 25 + \frac{4}{1+6\tau^2}$ .find mean and variance.

8. Define power spectral density.

Answer:

If $R_{XX}(\tau)$ is the ACF of a WSS process {X(t)} then the power spectral density $S_{XX}(\omega)$ of the process {X(t)}, is defined by

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau \quad \text{(or)} \quad S_{XX}(f) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i2\pi f\tau} d\tau$$

9. Express each of ACF and PSD of a stationary R.P in terms of the other.{(or) write down wiener khinchine relation }

Answer:

$R_{XX}(\tau)$ and $S_{XX}(\omega)$ are Fourier transform pairs.

i.e., $S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau$ and $R_{xx}(\tau) = \int_{-\infty}^{\infty} S_{XX}(\omega) e^{i\omega\tau} d\tau$

10. Define cross power spectral density of two random process {X(t)} and{Y(t)}.

Answer:

If {X(t)} and {Y(t)} are jointly stationary random processes with cross correlation function $R_{XY}(\tau)$,then cross power spectral density of {X(t)} and {Y(t)}is defined by

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-i\omega\tau} d\tau$$

11. State any two properties of power spectral density.

Answer:

i) $S(\omega) = S(-\omega)$

ii) $S(\omega) > 0$

iii)The spectral density of a process {X(t)},real or complex, is a real function of ω and non-negative.

12. If $R(\tau) = e^{-2\lambda|\tau|}$ is the ACF of a R.P{X(t)}, obtain the spectral density.

Answer:

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau = \int_{-\infty}^{\infty} e^{-2\lambda|\tau|} e^{-i\omega\tau} d\tau = 2\int_{0}^{\infty} e^{-2\lambda\tau} \cos\omega\tau \, d\tau = \frac{4\lambda}{4\lambda^2 + \omega^2}.$$

13. If $R_{YY}(\tau) = 2R_{XX}(\tau) - R_{XX}(\tau + 2a) - R_{XX}(\tau - 2a)$ is the ACF of Y(t)=X(t+a)-X(t-a), find $S_{YY}(\omega)$.

Answer:

Given $R_{YY}(\tau) = 2R_{XX}(\tau) - R_{XX}(\tau + 2a) - R_{XX}(\tau - 2a)$

Take Fourier transform on both sides,

$$\int_{-\infty}^{\infty} R_{YY}(\tau) e^{-i\omega\tau} d\tau = 2\int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau - \int_{-\infty}^{\infty} R_{XX}(\tau + 2a) e^{-i\omega\tau} d\tau - \int_{-\infty}^{\infty} R_{XX}(\tau - 2a) e^{-i\omega\tau} d\tau$$

$$S_{YY}(\omega) = 2S_{XX}(\omega) - \int_{-\infty}^{\infty} R_{XX}(\tau + 2a) e^{-i\omega\tau} d\tau - \int_{-\infty}^{\infty} R_{XX}(\tau - 2a) e^{-i\omega\tau} d\tau$$

$$\begin{aligned} &Let \quad \tau + 2a = u \quad and \quad \tau - 2a = v \\ &d\tau = du \qquad d\tau = dv \\ &\tau \to \infty \Rightarrow u \to \infty, \qquad \tau \to \infty \Rightarrow v \to \infty, \\ &\tau \to -\infty \Rightarrow u \to -\infty, \qquad \tau \to -\infty \Rightarrow v \to -\infty, \end{aligned}$$

$$S_{YY}(\omega) = 2S_{XX}(\omega) - \int_{-\infty}^{\infty} R_{XX}(u) e^{-i\omega(u-2a)} du - \int_{-\infty}^{\infty} R_{XX}(v) e^{-i\omega(v+2a)} dv$$

$$S_{YY}(\omega) = 2S_{XX}(\omega) - e^{i\omega 2a} S_{XX}(\omega) - e^{-i\omega 2a} S_{XX}(\omega)$$

$$S_{YY}(\omega) = 2S_{XX}(\omega) - 2\left(\frac{\left(e^{i\omega 2a} + e^{-i\omega 2a}\right)}{2}\right) S_{XX}(\omega)$$

$$S_{YY}(\omega) = 2S_{XX}(\omega) - 2\cos(2a\omega) S_{XX}(\omega) = 4\sin^2 a\omega S_{XX}(\omega).$$

14. The ACF of the random telegraph signal process is given by $R(\tau) = a^2 e^{-2\lambda|\tau|}$. Determine the power density spectrum of the random telegraph signal.

Answer:

Given $R(\tau) = a^2 e^{-2\lambda|\tau|}$

Power spectral density is

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} d\tau$$

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} d\tau = \frac{4a^2\lambda}{4\lambda^2 + \omega^2}.$$

15. Prove that the PSDF of a real WSS is twice the Fourier cosine transform of its ACF.

Answer:

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} d\tau = \int_{-\infty}^{\infty} R(\tau)\{\cos\omega\tau - \sin\omega\tau\} d\tau = 2\int_{0}^{\infty} R(\tau)\cos\omega\tau \, d\tau$$

= Fourier cosine transform of [2 $R(\tau)$].

16. Prove that the ACF of a real WSS is half the Fourier inverse cosine transform of its PSDF.

Answer:

$$R(\tau) = \int_{-\infty}^{\infty} S(\omega) e^{-i\omega\tau} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\{\cos\omega\tau + \sin\omega\tau\} d\omega = \frac{1}{\pi}\int_{0}^{\infty} S(\omega)\cos\omega\tau \, d\omega$$

= Fourier inverse cosine transform of [$\frac{1}{2} S(\omega)$].

17. State any four properties of cross power density spectrum.

Answer:

i) $S_{XY}(\omega) = S_{YX}(-\omega) = S_{YX}^{*}(\omega)$

ii) Re[$S_{XY}(\omega)$] and Re[$S_{YX}(\omega)$] are even function of ω

iii) Im[$S_{XY}(\omega)$] and Im[$S_{YX}(\omega)$] are odd function of ω

iv) $S_{XY}(\omega)$ =0and $S_{YX}(\omega)$ =0 if X(t) and Y(t) are orthogonal.

Unit V Liner Systems with Random Inputs

1. Define a system and Define the linear system.

Answer:

A system is a functional relationship between the input X(t) and the output Y(t).i.e., Y(t) =f[X(t)], $-\infty < t < \infty$.

A System is a functional relationship between the input X(t) and the output Y(t).

If f $[a_1X_1(t)+ a_2X_2(t)] = a_1 f[X_1(t)]+ a_2 f[X_2(t)]$, then f is called a linear system.

2. Define time invariant system.

Answer:

If Y(t+h) =f[X(t+h)] where Y(t) =f[X(t)],then f is called the time invariant system.

3. Define memoryless system.

Answer:

If the output $Y(t_1)$ at a given time $t=t_1$ depends only on $X(t_1)$ and not on any other past or future values of X(t),then the system is called as a memoryless system.

4. Define causal system.

Answer:

The value of the output $t=t_0$ depends only on the values of the input X(t) for $t\le t_0$

$Y(t_0)=f[X(t) \text{ for } t\le t_0]$

5. Define Stable system.

Answer:

If h(t) is absolutely integrable ,i.e., $\int_{-\infty}^{\infty}|h(t)|dt < \infty$,then the system is said to be stable in the sense that every bounded input gives bounded output.

If h(t)=0,when t<0the system is said to be causal.

6. Prove that Time Invariant System transfer function.

Answer:

If $X(\omega)$,$Y(\omega)$ and $H(\omega)$ are the Fourier transform of x(t),y(t) and h(t) respectively, then

$$Y(\omega)=X(\omega).H(\omega)$$

Proof:

$$y(\omega) = \int_{-\infty}^{\infty} y(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(u)h(t-u)du\right]e^{-i\omega t}dt = \int_{-\infty}^{\infty} x(u)\left[\int_{-\infty}^{\infty} h(t-u)e^{-i\omega(t-u)}dt\right]e^{-i\omega u}du$$

$$= \int_{-\infty}^{\infty} x(u)\,H(\omega)e^{-i\omega u}du = X(\omega).H(\omega)$$

7. Check whether the following system is linear .y(t)=t x(t)

Answer:

Consider two input functions $x_1(t)$ and $x_2(t)$. The corresponding outputs are

$y_1(t)=t\, x_1(t)$ and $y_2(t)=t\, x_2(t)$

Consider $y_3(t)$ as the linear combinations of the two inputs.

$y_3(t)= t[a_1 x_1(t)+a_2 x_2(t)]= a_1t\, x_1(t)+a_2\, t\, x_2(t)$ ……………..(1)

consider the linear combinations of the two outputs.

$a_1y(t)+a_2 y_2(t)= a_1t\, x_1(t)+a_2\, t\, x_2(t)$ …………………(2)

From (1)and(2), (1)=(2)

The system y(t)=t x(t) is linear.

8. Check whether the following system is linear .$y(t)= x^2(t)$

Answer:

Consider two input functions $x_1(t)$ and $x_2(t)$. The corresponding outputs are

$y_1(t)=x_1^2(t)$ and $y_2(t)=x_2^2(t)$

Consider $y_3(t)$ as the linear combinations of the two inputs.

$y_3(t)= [a_1 x_1(t)+a_2 x_2(t)]^2= a_1^2 x_1^2(t)+a_2^2 x_2^2(t)+2 a_1x_1(t)a_2x_2(t)$ ……..(1)

consider the linear combinations of the two outputs.

$a_1y(t)+a_2 y_2(t)= a_1 x_1^2(t)+a_2 x_2^2(t)$ …………………(2)

From (1) and (2), $(1) \neq (2)$

The system $y(t)=x^2(t)$ is not linear.

9. Check whether the following system is time-invariant $y(t)= x(t) – x(t-1)$

Answer:

Let the input alone be shifted in time by K units.

$y(t)= x(t-K) – x(t-K-1)$ ……………..(1)

Now consider that the output is shifted by K units.

$y(t-K)= x(t-K) – x(t-K-1)$ ……………….(2)

From (1) and (2), (1) =(2)

The given system is time invariant.

10. Check whether the following system is time-invariant $y(t)= t\,x(t)$

Answer:

Let the input alone be shifted in time by K units.

$y(t)=t\,x(t-K)$ ……………..(1)

Now consider that the output is shifted by K units.

$y(t-K)= (t-K)x(t-K) =tx(t-K)- Kx(t-K)$ ……………….(2)

From (1) and (2), $(1) \neq (2)$

The given system is not time invariant.

11. Define the Linear Time Invariant System.

Answer:

A linear system is said to be also time-invariant if the form of its impulse response $h(t,u)$ does not depend on the time that the impulse is applied.

For linear time invariant system, $h(t,u) = h(t-u)$

If a system is such that its Input X(t) and its Output Y(t) are related by a Convolution integral,

i.e., if $Y(t) = \int_{-\infty}^{\infty} h(u)\, X(t-u)\, du$, then the system is a linear time-invariant system.

12. Define Noise

The signal is not only distorted by the channel but also contaminated along the path by undesirable signals that are generally referred to by the term noise.

13.Define Thermal Noise.

Thermal noise is the noise because of the random motion of free electrons in conducting media such as a resister.Thermal noise generated in resisters and semiconducters is assumed to be zero mean, stationary Gaussian random process {N(t)} with a power spectral density.

14. Define White Noise.

A continuous time random process x(t) where $t \in R$ is a white noise iff if its mean function and auto correlation function satisfy the following :

$$\mu_x(t) = E\{x(t)\} = 0$$

$$R_{xx}(t_1 t_2) = E\{x(t_1)x(t_2)\} = (\frac{N_0}{2}\delta(t_1 - t_2))$$

i.e., it is a zero mean process for all time and has infinite power at zero time shift since its autocorrelation function is the Dirac delta function.

15. Find the ACF of the random process {X(t)}, if its power spectral density is given by

$$S(\omega) = \begin{cases} 1+\omega^2, & for\ |\omega| \le 1 \\ 0 & , for\ |\omega| > 1 \end{cases}$$

Solution:

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)e^{i\omega\tau}\, d\omega = \frac{1}{2\pi}\int_{-1}^{1}\{1+\omega^2\}e^{i\omega\tau} d\omega = \frac{1}{2\pi}\int_{-1}^{1}\left\{e^{i\omega\tau} + \omega^2 e^{i\omega\tau}\right\}d\omega = \frac{1}{2\pi}\left\{\left[\frac{e^{i\omega\tau}}{i\tau}\right]_{-1}^{1} + \int_{-1}^{1}\omega^2 \cos\omega\tau\ d\omega\right\}$$

$$= \frac{1}{2\pi}\left\{\left[\frac{e^{i\omega\tau}}{i\tau}\right]_{-}^{1} + 2\int_{0}^{1}\omega^2 \cos\omega\tau\ d\omega\right\} = \frac{1}{2\pi}\left[\frac{e^{i\tau} - e^{-i\tau}}{i\tau}\right] + 2\left[\omega^2\frac{\sin\omega\tau}{\tau} + \frac{2\omega\cos\omega\tau}{\tau^2} - \frac{2\sin\omega\tau}{\tau^3}\right]_0^1$$

$$= \frac{1}{2\pi}\left[\frac{2\sin\tau}{\tau} + \frac{2\sin\tau}{\tau} + \frac{4\cos\tau}{\tau^2} - \frac{4\sin\tau}{\tau^3}\right] = \frac{1}{2\pi}\left[\frac{2\tau^2\sin\tau + 2\tau^2\sin\tau + \tau 4\cos\tau - 4\sin\tau}{\tau^3}\right]$$

$$= \frac{2\{\tau^2\sin\tau + \tau\cos\tau - \sin\tau\}}{\pi\tau^3}$$

16. Define Average power.

$$\text{Average power } = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\ d\omega = R(0)$$